MONTE CARLO AND MOMENT ESTIMATION FOR PARAMETERS OF A BLACK SCHOLES MODEL FROM AN INFORMATION-BASED PERSPECTIVE (THE BS-BHM MODEL):A COMPARISON WITH THE BS-BHM UPDATED MODEL

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Abstract. This paper presents estimation of parameters on the BS-BHM model by using Monte Carlo and Moments estimate as they have been done in BS-BHM Updated model. BS-BHM Updated model is BS-BHM model that it is improved the result of Gaussian integral, especially in completing square. Estimation of parameters use Monte Carlo and moments estimate under BS-BHM model results the equation of polynomial of four degree . While estimation of parameters under BS-BHM Updated model results the quadratic equation. Application for real data of Microsoft shares (MSFT), under BS-BHM model results four different estimates values, while under BS-BHM Updated model results one estimate value.

Key words and Phrases : BS-BHM model, Monte Carlo estimate, Moment estimate,and Comparison.

1. Introduction

Black Scholes asset pricing model from an information-based perspective has been developed by Brody Hughston Macrina (BHM). It is developed from a special condition of asset pricing model from an information-based approaches by Brody Hughston Macrina (BHM model or BHM approach). Further, Black Scholes model from an information-based perspective is called BS-BHM model in this paper. Explicitly, BS-BHM model is presented, see Macrina, A.[6]

$$S_t = P_{tT}\, S_0 \exp\left(rT - \frac{1}{2}\nu^2 T + \frac{1}{2}\frac{\nu\sqrt{T}}{\sigma^2\tau + 1} + \frac{\sigma\tau\,\nu\sqrt{T}}{t(\sigma^2\tau + 1)}\,\xi_t\right) \tag{1.1}$$

or

$$S_t = S_0 \exp\left(rt - \frac{1}{2}\nu^2 T + \frac{1}{2}\frac{\nu\sqrt{T}}{\sigma^2\tau + 1} + \frac{\sigma\tau\,\nu\sqrt{T}}{t(\sigma^2\tau + 1)}\,\xi_t\right) \tag{1.2}$$

or

$$\frac{S_t}{S_0} = \exp\left(rt - \frac{1}{2}v^2T + \frac{1}{2}\frac{v\sqrt{T}}{\sigma^2\tau+1} + \frac{\sigma\tau\, v\sqrt{T}}{t(\sigma^2\tau+1)}\,\xi_t\right) \tag{1.3}$$

The BS-BHM model in equation (1.2) which it is equal to equation (1.3) has two parameters i.e. the asset price volatility parameter ν and the information flow rate parameter σ. It can be showed that the random variable of $\frac{S_t}{S_0}$ has lognormal distribution with density function

$$f\left(\frac{S_t}{S_0}\right) = \frac{1}{\frac{S_t}{S_0}\sqrt{2\pi}\; b\sqrt{\sigma^2t^2+\frac{t(T-t)}{T}}}\exp\left\{-\frac{1}{2}\frac{\left(\log\frac{S_t}{S_0}-A\right)^2}{b^2\left(\sigma^2t^2+\frac{t(T-t)}{T}\right)}\right\} \tag{1.4}$$

or

$$f\left(\frac{S_t}{S_0}\right) = \frac{1}{\frac{S_t}{S_0}\sqrt{2\pi}\; B}\exp\left\{-\frac{1}{2}\frac{\left(\log\frac{S_t}{S_0}-A\right)^2}{B^2}\right\} \tag{1.5}$$

where $A = rt - \frac{1}{2}v^2T + \frac{1}{2}\frac{v\sqrt{T}}{\sigma^2\tau+1}$ and

$$B^2 = b^2\left(\sigma^2t^2+\frac{t(T-t)}{T}\right) = \left(\frac{\sigma\tau\, v\sqrt{T}}{t(\sigma^2\tau+1)}\right)^2\left(\sigma^2t^2+\frac{t(T-t)}{T}\right)$$

In other words, random variable of $\log\frac{S_t}{S_0}$ is normally distributed with mean is $A = rt - \frac{1}{2}v^2T + \frac{1}{2}\frac{v\sqrt{T}}{\sigma^2\tau+1}$ and variance is $B^2 = \left(\frac{\sigma\tau\, v\sqrt{T}}{t(\sigma^2\tau+1)}\right)^2\left(\sigma^2t^2+\frac{t(T-t)}{T}\right)$. Further, it is written by $\log\frac{S_t}{S_0} \sim N(A, B^2)$.

The BS-BHM model as in equation (1.2) is derived from BHM model by a special condition. BHM model is built for case of cash flow is payout of the associated dividend of equity. Further, the process of deriving of BS-BHM model as in Macrina,A [6] and Mutijah, Guritno, S. and Gunardi [7].

Explicitly, the asset pricing model is presented as follows,

$$S_t = P_{tT}E^{\mathbb{Q}}[D_T|\mathcal{F}_t] \tag{1.6}$$

S_t is the value of cash flows at time t, $0 \le t < T$ from asset that payout single dividend D_T at time T . In equation (1.6), P_{tT} represents the discount factors that it is to be equal to $e^{-r(T-t)}$ with r is the interest rates. Then $\mathbb{Q}$ is the risk neutral probability, and $\mathcal{F}_t$ is the market information filtration.

Modeling the infornation flows is based on an assumption that the information about dividends which is available in market is contained by the process $\{\xi_t\}_{0\le t\le T}$ defined by :

$$\xi_t = \sigma t D_T + \beta_{tT} \tag{1.7}$$

$\{\xi_t\}$ is a market information process. The market information process is composed from two parts, they are σtD_T which refers to the true information about dividends and $\{\beta_{tT}\}_{0\le t\le T}$ which refers to a standard Brownian Bridge on interval [0, T]. In the formula of asset pricing model by Brody Hughston Macrina in equation (1.6) above, if random variable D_T is equal to x which it has continuous distribution then ,

$$E^{\mathbb{Q}}[D_T|\mathcal{F}_t] = E^{\mathbb{Q}}[D_T|\,\xi_t] = \int_0^\infty x\pi_t(x)\,dx \tag{1.8}$$

where

$$\pi_t(x) = \frac{d}{dx}\,\mathbb{Q}(D_T \le x|\xi_t) \tag{1.9}$$

By using Bayes formula in Box-Tiao [2], $\pi_t(x)$ is presented in Brody, D.C, Hughston, L.P, and Macrina, A [1], Caliskan, N [3], Macrina,A [6] and Mutijah, Guritno, S. and Gunardi [7] as follows

$$\pi_t(x) = \frac{p(x)\rho(\xi_t|D_T=x)}{\rho(\xi_t)} \qquad (1.10)$$

and the final result of the BHM model or the BHM approach,

$$S_t = P_{tT} \frac{\int_0^\infty x\, p(x) \exp\left(\frac{T}{T-t}\left(\sigma x\xi_t - \frac{1}{2}\sigma^2 x^2 t\right)\right) dx}{\int_0^\infty p(x)\exp\left(\frac{T}{T-t}\left(\sigma x\xi_t - \frac{1}{2}\sigma^2 x^2 t\right)\right) dx} \qquad (1.11)$$

Brody Hughston Macrina also built the other concept for the asset pricing model that it is derived from the formula of equation (1.6) for a special condition where it is a limited-liability asset which pays no interim dividends and at time T it is sold off for the value S_T. S_T is log-normally distributed and has the form of

$$S_T = S_0 \exp(rT - \frac{1}{2}\, v^2\, T + v\sqrt{T}\, X_T) \qquad (1.12)$$

where S_0, r, $\boldsymbol{v}$ are given constants and X_T is a standard normally distributed random variable. The corresponding information process is given by

$$\xi_t = \sigma t X_T + \beta_{tT} \qquad (1.13)$$

The price proses $\{S_t\}_{0\leq t\leq T}$ is obtained from :

$$S_t = P_{tT}\, E^{\mathbb{Q}}(\Delta_T(X_T)|\, \xi_t) \qquad (1.14)$$

Then for $t < T$, the equation S_t results :

$$S_t = P_{tT} \int_{-\infty}^{\infty} \Delta_T(x)\, \pi_{tT}(x)\, dx \qquad (1.15)$$

And by the Bayes formula, it is obtained $\pi_{tT}(x)$ as follows

$$\pi_{tT}(x) = \frac{p(x) \exp\left[\frac{T}{T-t}\left(\sigma x\xi_t - \frac{1}{2}\sigma^2 x^2 t\right)\right]}{\int_{-\infty}^{\infty} p(x) \exp\left[\frac{T}{T-t}\left(\sigma x\xi_t - \frac{1}{2}\sigma^2 x^2 t\right)\right] dx}$$

(1.16)

In this case, S_T plays the role of single cash flow $\Delta_T(x)$ for $X_T = x$.

So, it is obtained the equation S_t as follows

$$S_t = P_{tT} \int_{-\infty}^{\infty} S_0 \exp\left(rT - \frac{1}{2}\, \nu^2 T + \nu\sqrt{T}\, x\right) \frac{p(x) \exp\left[\frac{T}{T-t}\left(\sigma x\xi_t - \frac{1}{2}\sigma^2 x^2 t\right)\right]}{\int_{-\infty}^{\infty} p(x) \exp\left[\frac{T}{T-t}\left(\sigma x\xi_t - \frac{1}{2}\sigma^2 x^2 t\right)\right] dx} dx \quad (1.17)$$

Because X_T is assumed to be standard normally distributed then

$$p(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}\, x^2) \qquad (1.18)$$

To follow the Gaussian integrals in Macrina, A [6] and Straub, W.O. [12] then S_t becomes

$$S_t = P_{tT}\, S_0 \exp\left(rT - \frac{1}{2}\, \nu^2 T\right) \frac{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-\frac{1}{2}x^2) \exp\left[\left(\frac{T}{T-t}\sigma\xi_t + \nu\sqrt{T}\right)x - \frac{1}{2}\, \frac{T}{T-t}\sigma^2 t x^2\right] dx}{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-\frac{1}{2}x^2) \exp\left[\frac{T}{T-t}\sigma\xi_t x - \frac{1}{2}\, \frac{T}{T-t}\sigma^2 t x^2\right] dx}$$

(1.19)

By using Gaussian integrals, the equation of asset pricing model S_t is given below

$$S_t = S_0 \exp\left(rt - \frac{1}{2}\, \frac{\sigma^2\tau}{\sigma^2\tau + 1}\nu^2 T + \frac{\sigma\tau\, \nu\sqrt{T}}{t(\sigma^2\tau + 1)}\, \xi_t\right)$$

(1.20)

where $\tau = \frac{tT}{(T-t)}$. Successive steps to obtain the model in equation (1.20) can be seen in Mutijah, Guritno, S. and Gunardi [7]. Furthermore, the model in equation (1.20) is called the BS-BHM Updated model.

In BS-BHM model, there are also the asset price volatility parameter ν and

true information flow rate parameter σ which they can not be observed directly. This paper will discuss the two parameter estimation for the previous behaviour of asset price. The estimation value of parameter ν and σ arisen from this general procedure is called a historical volatility and information flow rate estimation.
The deriving of BHM model with a special condition which is built by Brody Hughston Macrina , it must be as in equation (1.20).Therefore, this paper will compare the results of parameter estimation between estimation of parameters and its application for real data of Microsoft shares (MSFT) under the BS-BHM model with under the BS-BHM Updated model. Hence, estimation of parameters and its application for real data of Microsoft shares (MSFT) under the BS-BHM Updated model can be seen in Mutijah, Guritno, S. and Gunardi [8].

2. Main Results

2.1. Monte Carlo Estimate

Estimation procedure of Monte Carlo under the BS-BHM the same as estimation procedure of Monte Carlo under the BS-BHM Updated model. They are based on procedure by Higham,D.J. [5] as follows

Suppose that historical asset price data is available at equally spaced time values $t_i = i\ \Delta t$,so S_{t_i} is the asset price at time t_i . Defined $U_i = \log \frac{S_{t_i}}{S_{t_{i-1}}}$ and $\{U_i\}$ are independent as in Higham, D.J. [5]. To estimate the asset price volatility ν and the information flow rate σ on the BS-BHM model by using Monte Carlo approach that it is written Higham, D.J. [5] as follows :

Suppose that $t = t_n$ is the current time and that the M+1 is most current asset prices. $\left\{S_{t_{n-M}},\ S_{t_{n-M+1}}, \dots\ S_{t_{n-1}}, S_{t_n}\right\}$ is also available and by using the corresponding log rasio data which is $\left\{U_{n+1-i}\right\}_{i=1}^{M}$, then the sample mean dan variance estimation are

$$a_M = \frac{1}{M} \sum_{i=1}^{M} U_{n+1-i} \tag{2.1}$$

and

$$b_M^2 = \frac{1}{M-1} \sum_{i=1}^{M} \left(U_{n+1-i} - a_M\right)^2 \tag{2.2}$$

Monte Carlo estimate is done by comparing the sample mean with the mean of BS-BHM model or by comparing the sample variance with the variance of BS-BHM model as below

$$a_M = rt - \frac{1}{2}\nu^2 T + \frac{1}{2}\frac{\nu\sqrt{T}}{\sigma^2\tau + 1}$$

or

$$\nu^2 - \frac{1}{\sqrt{T}\,(\sigma^2\tau + 1)}\nu + \frac{2\,(a_M - rt)}{T} = 0 \tag{2.3}$$

and

$$b_M^2 = \left(\frac{\sigma\tau\,\nu\sqrt{T}}{t(\sigma^2\tau + 1)}\right)^2 \left(\sigma^2 t^2 + \frac{t(T-t)}{T}\right)$$
$$= \left(\frac{\sigma\tau\sqrt{T}}{t(\sigma^2\tau + 1)}\right)^2 \left(\sigma^2 t^2 + \frac{t(T-t)}{T}\right)\nu^2$$

or

$$\nu^2 = \frac{b_M^2}{\left(\frac{\sigma\tau\sqrt{T}}{t(\sigma^2\tau + 1)}\right)^2 \left(\sigma^2 t^2 + \frac{t(T-t)}{T}\right)} \tag{2.4}$$

Substitution equation (2.4) to equation (2.3) is obtained by algebra tricks successively as below

$$\frac{b_M^2}{\left(\frac{\sigma\tau\sqrt{T}}{t(\sigma^2\tau+1)}\right)^2\left(\sigma^2t^2+\frac{t(T\text{-}t)}{T}\right)} - \frac{1}{\sqrt{T}\,(\sigma^2\tau+1)}\cdot\frac{\sqrt{b_M^2}}{\left(\frac{\sigma\tau\sqrt{T}}{t(\sigma^2\tau+1)}\right)\sqrt{\sigma^2t^2+\frac{t(T\text{-}t)}{T}}} + \frac{2\,(a_M - rt)}{T} = 0$$

Making the same in denumerator to the equation is obtained

$$\frac{b_M^2\,t^2(\sigma^2\tau+1)^2 - \sigma\tau t\sqrt{b_M^2}\,\sqrt{\sigma^2t^2+\frac{t(T\text{-}t)}{T}} + 2\,(a_M - rt)(\sigma\tau\sqrt{T})^2\left(\sigma^2t^2+\frac{t(T\text{-}t)}{T}\right)}{(\sigma\tau\sqrt{T})^2\left(\sigma^2t^2+\frac{t(T\text{-}t)}{T}\right)} = 0$$

By dividing to multiplying then

$$b_M^2\,t^2(\sigma^2\tau+1)^2 - \sigma\tau t\sqrt{b_M^2}\,\sqrt{\sigma^2t^2+\frac{t(T\text{-}t)}{T}} + 2\,(a_M - rt)(\sigma\tau\sqrt{T})^2\left(\sigma^2t^2+\frac{t(T\text{-}t)}{T}\right) = 0$$

Changing in form of equation the same as

$$\sigma\tau t\sqrt{b_M^2}\,\sqrt{\sigma^2t^2+\frac{t(T\text{-}t)}{T}} = b_M^2\,t^2(\sigma^2\tau+1)^2 + 2\,(a_M - rt)(\sigma\tau\sqrt{T})^2\left(\sigma^2t^2+\frac{t(T-t)}{T}\right)$$

And by multiplying to dividing is obtained

$$\sqrt{\sigma^2t^2+\frac{t(T\text{-}t)}{T}} = \frac{\sqrt{b_M^2}\;t\,(\sigma^2\tau+1)^2}{\sigma\tau} + \frac{2\,(a_M - rt)(\sigma\tau\sqrt{T})^2\left(\sigma^2t^2+\frac{t(T-t)}{T}\right)}{t\sqrt{b_M^2}}$$

The left and right hands are squared

$$\sigma^2t^2+\frac{t(T\text{-}t)}{T} = \frac{b_M^2\,t^2(\sigma^2\tau+1)^4}{\sigma^2\tau^2} + 4T\,(a_M - rt)\left(\sigma^2t^2+\frac{t(T-t)}{T}\right)(\sigma^2\tau+1)^2 + \frac{4\,(a_M - rt)^2(\sigma^2\tau^2T^2)\left(\sigma^2t^2+\frac{t(T-t)}{T}\right)^2}{t^2\,b_M^2}$$

Thus it is made in equation is to be equal to zero

$$\frac{b_M^2\,t^2(\sigma^2\tau+1)^4}{\sigma^2\tau^2} + 4T(a_M - rt)\left(\sigma^2\,t^2+\frac{t(T-t)}{T}\right)(\sigma^2\tau+1)^2 + \frac{4\,(a_M - rt)^2(\sigma^2\tau^2T^2)\left(\sigma^2t^2+\frac{t(T-t)}{T}\right)^2}{t^2\,b_M^2} - \sigma^2t^2 - \frac{t(T\text{-}t)}{T} = 0$$

Successively, by algebra tricks then it is held the final result is the polynomial equation as below

$$A\sigma^8 + B\sigma^6 + C\sigma^4 + D\sigma^2 + E = 0$$

where

$$A = b_M^2\,t^2\tau^2 + 4T\,(a_M - rt)\,t^2\,\tau^2 + \frac{4\,(a_M - rt)^2\,\tau^2T^2t^2}{b_M^2}$$

$$B = b_M^2\,t^2\tau + 4T\,(a_M - rt)\left(2t^2\tau + \frac{t\,(T-t)\,\tau^2}{T}\right) + \frac{8\,(a_M - rt)^2\,\tau^2\,t\,T(T\text{-}t)}{b_M^2}$$

$$C = 6\,b_M^2\,t^2 + 4T\,(a_M - rt)\left(t^2 + \frac{t\,(T-t)\,\tau}{T}\right) + \frac{4\,(a_M - rt)^2\,\tau^2\,(T\text{-}t)^2}{b_M^2}$$

$$D = \frac{4\,b_M^2\,t^2}{\tau} + 4(a_M - rt)t(T-t) - \frac{t\,(T-t)}{T}$$

$$E = \frac{b_M^2\,t^2}{\tau^2}$$

Let $\sigma^2 = x$, then it is obtained the polynomial of four degree

$$Ax^4 + Bx^3 + Cx^2 + Dx + E = 0$$

The equation of polynomial of four degree has four solution in $x = \sigma^2$.

Further, to determine estimation of parameter ν then the solutions in $x = \sigma^2$ are substituted to equation (2.3) that is

$$\nu^2 - \frac{1}{\sqrt{T}\,(\sigma^2\tau + 1)}\nu + \frac{2\,(a_M - rt)}{T} = 0$$

Because equation (2.3) has solution

$$\nu = \frac{1}{2\sqrt{T}\,(\sigma^2\tau + 1)} + \frac{1}{2}\sqrt{\left(-\frac{1}{\sqrt{T}\,(\sigma^2\tau + 1)}\right)^2 - 4.1.\frac{2\,(a_M - rt)}{T}}$$

or

$$\nu = \frac{1}{2\sqrt{T}\,(\sigma^2\tau + 1)} - \frac{1}{2}\sqrt{\left(-\frac{1}{\sqrt{T}\,(\sigma^2\tau + 1)}\right)^2 - 4.1.\frac{2\,(a_M - rt)}{T}}$$

so to determine estimation of parameter ν then the solutions in $x = \sigma^2$ also can be substituted to two solution above. Finally, the estimate of parameters ν and σ are held.

2.2. Moment Estimate

Analogous to Monte Carlo estimate, suppose that historical asset price data is available at equally spaced time values $t_i = i\,\Delta t$,so S_{t_i} is the asset price at time t_i. Defined $U_i = \log \frac{S_{t_i}}{S_{t_{i-1}}}$ and $\{U_i\}$ are independent. Parameters estimation of asset price volatility ν and the information flow rate σ of BS-BHM model using the method of moment as follows

Suppose that $t = t_n$ is the current time and that the M+1 is most current asset prices. $\{S_{t_{n-M}}, S_{t_{n-M+1}}, \ldots S_{t_{n-1}}, S_{t_n}\}$ is also available and by using the corresponding log rasio data which is $\{U_{n+1-i}\}_{i=1}^{M}$ then the first sample moment (m_1 = mean) and the second sample moment (m_2), see Higham, D.J.[5], Mutijah, Guritno, S., and Gunardi [8], Shao, J. [11], and Subanar [13] are

$$m_1 = \frac{1}{M}\sum_{i=1}^{M} U_{n+1-i} \tag{2.5}$$

and

$$m_2 = \frac{1}{M}\sum_{i=1}^{M} \left(U_{n+1-i}\right)^2 \tag{2.6}$$

Estimation of parameters in moment estimate can be obtained by making the k-th moment of the sample to be equal to the k-th moment of the model.

Suppose $\mu_1$dan μ_2 are the first moment and the second moment for BS-BHM model, then

$$\mu_1 = E(U_i) = rt - \frac{1}{2}\nu^2 T + \frac{1}{2}\frac{\nu\sqrt{T}}{\sigma^2\tau + 1} \tag{2.7}$$

and

$$\begin{aligned}\mu_2 = E\left(U_i^2\right) &= \mathrm{Var}(U_i) + \left(E(U_i)\right)^2 \\ &= \left(\frac{\sigma\tau\,\nu\sqrt{T}}{t(\sigma^2\tau + 1)}\right)^2\left(\sigma^2 t^2 + \frac{t(T-t)}{T}\right) - \left(rt - \frac{1}{2}\nu^2 T + \frac{1}{2}\frac{\nu\sqrt{T}}{\sigma^2\tau + 1}\right)^2\end{aligned} \tag{2.8}$$

From equation (2.5) and equation (2.7) are obtained equation belows

$$m_1 = rt - \frac{1}{2}\nu^2 T + \frac{1}{2}\frac{\nu\sqrt{T}}{\sigma^2\tau + 1}$$

or

$$\nu^2 - \frac{1}{\sqrt{T}\,(\sigma^2\tau + 1)}\,\nu + \frac{2\,(m_1 - rt)}{T} = 0 \tag{2.9}$$

and from equation (2.6) and equation (2.8) are also obtained equation

$$m_2 = \left(\frac{\sigma\tau\, v\sqrt{T}}{t(\sigma^2\tau+1)}\right)^2\left(\sigma^2 t^2 + \frac{t(T-t)}{T}\right) - \left(rt - \frac{1}{2}v^2 T + \frac{1}{2}\frac{v\sqrt{T}}{\sigma^2\tau+1}\right)^2$$

$$= \left(\frac{\sigma\tau\sqrt{T}}{t(\sigma^2\tau+1)}\right)^2\left(\sigma^2 t^2 + \frac{t(T-t)}{T}\right)v^2 - (m_1)^2$$

or

$$v^2 = \frac{m_2+(m_1)^2}{\left(\frac{\sigma\tau\sqrt{T}}{t(\sigma^2\tau+1)}\right)^2\left(\sigma^2 t^2+\frac{t(T-t)}{T}\right)} = \frac{m_2+(m_1)^2}{\frac{\sigma^4\tau^2 T}{(\sigma^2\tau+1)^2}+\frac{\sigma^2\tau^2(T-t)}{t(\sigma^2\tau+1)^2}} = \frac{m_2+(m_1)^2}{\frac{\sigma^4\tau^2 tT+\sigma^2\tau^2(T-t)}{t(\sigma^2\tau+1)^2}}$$

$$= \frac{t(\sigma^2\tau+1)^2(m_2+(m_1)^2)}{\sigma^4\tau^2 tT+\sigma^2\tau^2(T-t)}$$

(2.10)

Substitution of equation (2.10) to equation (2.9) is obtained

$$\frac{m_2+(m_1)^2}{\left(\frac{\sigma\tau\sqrt{T}}{t(\sigma^2\tau+1)}\right)^2\left(\sigma^2t^2+\frac{t(T-t)}{T}\right)} - \frac{1}{\sqrt{T}\,(\sigma^2\tau+1)}\cdot\frac{\sqrt{m_2+(m_1)^2}}{\left(\frac{\sigma\tau\sqrt{T}}{t(\sigma^2\tau+1)}\right)\sqrt{\sigma^2t^2+\frac{t(T-t)}{T}}} + \frac{2(m_1-rt)}{T} = 0$$

Making the same in denumerator to the equation is obtained

$$\frac{(m_2+(m_1)^2)\,t^2(\sigma^2\tau+1)^2 - \sigma\tau t\sqrt{m_2+(m_1)^2}\,\sqrt{\sigma^2t^2+\frac{t(T-t)}{T}} + 2(m_1-rt)(\sigma\tau\sqrt{T})^2\left(\sigma^2t^2+\frac{t(T-t)}{T}\right)}{(\sigma\tau\sqrt{T})^2\left(\sigma^2t^2+\frac{t(T-t)}{T}\right)} = 0$$

By dividing to multiplying then $(m_2 + (m_1)^2)\, t^2(\sigma^2\tau + 1)^2 - \sigma\tau t$ $\sqrt{m_2 + (m_1)^2}\,\sqrt{\sigma^2 t^2 + \frac{t(T-t)}{T}} +$

$$2(m_1 - rt)(\sigma\tau\sqrt{T})^2\left(\sigma^2t^2 + \frac{t(T-t)}{T}\right) = 0$$

Changing in form of equation the same as

$$\sigma\tau t\sqrt{m_2 + (m_1)^2}\,\sqrt{\sigma^2t^2 + \frac{t(T-t)}{T}} = (m_2 + (m_1)^2)t^2(\sigma^2\tau + 1)^2 + 2(m_1 - rt)(\sigma\tau\sqrt{T})^2\left(\sigma^2t^2 + \frac{t(T-t)}{T}\right)$$

and by multiplying to dividing is obtained

$$\sqrt{\sigma^2t^2+\frac{t(T-t)}{T}} = \frac{\sqrt{m_2+(m_1)^2}\;t\,(\sigma^2\tau+1)^2}{\sigma\tau} + \frac{2(m_1-rt)(\sigma\tau\sqrt{T})^2\left(\sigma^2t^2+\frac{t(T-t)}{T}\right)}{t\sqrt{m_2+(m_1)^2}}$$

The left and right hands are squared

$$\sigma^2t^2+\frac{t(T-t)}{T} = \frac{m_2+(m_1)^2\,t^2(\sigma^2\tau+1)^4}{\sigma^2\tau^2} + 4T(m_1 - rt)\left(\sigma^2t^2 + \frac{t(T-t)}{T}\right)(\sigma^2\tau+1)^2 + \frac{4(m_1-rt)^2(\sigma^2\tau^2T^2)\left(\sigma^2t^2+\frac{t(T-t)}{T}\right)^2}{t^2\,(m_2+(m_1)^2)}$$

Thus, it is made in equation is to be equal to zero

$$\frac{(m_2 + (m_1)^2)\,t^2(\sigma^2\tau + 1)^4}{\sigma^2\tau^2} + 4T(m_1 - rt)\left(\sigma^2 t^2 + \frac{t(T-t)}{T}\right)(\sigma^2\tau+1)^2 + \frac{4(m_1 - rt)^2(\sigma^2\tau^2T^2)\left(\sigma^2t^2 + \frac{t(T-t)}{T}\right)^2}{t^2\,(m_2 + (m_1)^2)} - \sigma^2t^2 - \frac{t(T-t)}{T} = 0$$

Successively, by algebra tricks then it is held the final result is the polynomial equation as below

$$A\sigma^8 + B\sigma^6 + C\sigma^4 + D\sigma^2 + E = 0$$

where

$$A = (m_2 + (m_1)^2)\,t^2\tau^2 + 4T(m_1 - rt)\,t^2\tau^2 + \frac{4(m_1-rt)^2\,\tau^2T^2t^2}{m_2+(m_1)^2}$$

$$B = (m_2 + (m_1)^2)\, t^2\tau + 4T(m_1 - rt)\left(2t^2\tau + \frac{t\,(T-t)\,\tau^2}{T}\right) + \frac{8(m_1 - rt)^2\,\tau^2\, t\, T(T\text{-}t)}{m_2 + (m_1)^2}$$

$$C = 6\,(m_2 + (m_1)^2)\, t^2 + 4T(m_1 - rt)\left(t^2 + \frac{t\,(T-t)\,\tau}{T}\right) + \frac{4(m_1 - rt)^2\,\tau^2\,(T\text{-}t)^2}{m_2 + (m_1)^2}$$

$$D = \frac{4\left(m_2 + (m_1)^2\right) t^2}{\tau} + 4(m_1 - rt)t(T-t) - \frac{t\,(T-t)}{T}$$

$$E = \frac{\left(m_2 + (m_1)^2\right) t^2}{\tau^2}$$

Let $\sigma^2 = x$, then it is obtained the polynomial of four degree

$$Ax^4 + Bx^3 + Cx^2 + Dx + E = 0$$

The equation of polynomial of four degree has four solution in $x = \sigma^2$.

Further, to determine estimation of parameter ν then the solutions in $x = \sigma^2$ are substituted to equation (2.9) that is

$$\nu^2 - \frac{1}{\sqrt{T}\,(\sigma^2\tau + 1)}\nu + \frac{2\,(a_M - rt)}{T} = 0$$

Because equation (2.9) has solution

$$\nu = \frac{1}{2\sqrt{T}\,(\sigma^2\tau + 1)} + \frac{1}{2}\sqrt{\left(-\frac{1}{\sqrt{T}\,(\sigma^2\tau + 1)}\right)^2 - 4.1.\frac{2\,(a_M - rt)}{T}}$$

or

$$\nu = \frac{1}{2\sqrt{T}\,(\sigma^2\tau + 1)} - \frac{1}{2}\sqrt{\left(-\frac{1}{\sqrt{T}\,(\sigma^2\tau + 1)}\right)^2 - 4.1.\frac{2\,(a_M - rt)}{T}}$$

so to determine estimation of parameter ν then the solutions in $x = \sigma^2$ also can be substituted to the two solution above. Finally, estimation of parameters ν and σ are held too.

2.3. Application for Example of Real Data

Under BS-BHM and BS-BHM Updated model, estimation of historical volatility and the information flow rate of Microsoft (MSFT) shares for Monthly data are done by using Monte Carlo and moment estimates. For two parameters and two methods of estimates, it is assumed that the data corresponds to equally spaced points in time as in Higham, D.J [5], and Mutijah, Guritno, S., and Gunardi [8].

In Monte Carlo estimate, the monthly data runs over 5 years (T = 5 years) and has 60 asset prices (M = 59), so it has dt = T/M = 5/59 ≈ 0,084746. For the monthly data result in $a_M = -1{,}47 \times 10^{-3}$ and $b_M^2 = 1{,}056 \times 10^{-3}$. Under BS-BHM model, estimation based on Monte Carlo produces estimation of parameters ν and σ , they are

$\hat{\nu}_1 = 3.1822$, $\hat{\sigma}_1 = 3.9882$,
$\hat{\nu}_2 = -0.0796 + 0.6060i$, $\hat{\sigma}_2 = 1.0545 + 3.7081i$,
$\hat{\nu}_3 = -0.0796 - 0.6060i$, $\hat{\sigma}_3 = 1.0545 - 3.7081i$,
$\hat{\nu}_4 = 1.3451$, $\hat{\sigma}_4 = 0.1084$
$\hat{\nu}_5 = 2.1200$, $\hat{\sigma}_1 = 3.9882$,
$\hat{\nu}_6 = -0.1205 + 0.9014i$, $\hat{\sigma}_2 = -0.1205 + 0.9014i$,
$\hat{\nu}_7 = -0.1205 - 0.9014i$, $\hat{\sigma}_3 = 1.0545 - 3.7081i$,
$\hat{\nu}_8 = 0.8933$, $\hat{\sigma}_4 = 0.1084$

There are eight estimation of parameters ν and σ. They include four real number and four imaginary number. Therefore, estimation of parameters ν and σ are chosen four real number.

In moment estimate, the monthly data runs over 5 years (T = 5 years) and has 60 asset prices (M = 59), so it has dt = T/M = 5/59 ≈ 0,084746. For the monthly data result in $m_1 = -1{,}47 \times 10^{-3}$ and $m_2 = 1{,}1 \times 10^{-3}$. Under BS-BHM model, estimation based on the moment estimate produces estimation of parameters ν and σ, they are

$\hat{\nu}_1 = 3.1849$, $\hat{\sigma}_1 = 3.9911$,
$\hat{\nu}_2 = -0.0803 + 0.6057i$, $\hat{\sigma}_2 = 1.0537 + 3.7091i$,
$\hat{\nu}_3 = -0.0803 - 0.6057i$, $\hat{\sigma}_3 = -0.0803 - 0.6057i$,
$\hat{\nu}_4 = 1.3451$, $\hat{\sigma}_4 = 0.1085$
$\hat{\nu}_5 = 2.1218$, $\hat{\sigma}_1 = 3.9911$,
$\hat{\nu}_6 = -0.1215 + 0.9010i$, $\hat{\sigma}_2 = 1.0537 + 3.7091i$,
$\hat{\nu}_7 = -0.1215 - 0.9010i$, $\hat{\sigma}_3 = -0.0803 - 0.6057i$,
$\hat{\nu}_8 = 0.8933$, $\hat{\sigma}_4 = 0.1085$

The same as Monte Carlo estimate, there are eight estimation of parameters ν and σ too. They also include four real number and four imaginary number. Therefore, estimation of parameters ν and σ are chosen four real number too.

Whereas, to compare the results of estimation parameters under BS-BHM model with under BS-BHM Updated model, it can be seen in Mutijah, Guritno,S, and Gunardi [8].

3. Concluding Remarks

BHM model or BHM approach is modelled by Brody Hughston Macrina (BHM) under the assumption that market participants do not have acces to the information about the actual value of the relevant market variable. Brody Hughston Macrina defined an asset by its cash flow structure and then the associated market factor is the upcoming cash flow that is the upcoming dividend. Brody Hughston Macrina also built BHM model by a special condition which it is called Black Scholes model from an information-based perspective or it is called BS-BHM model in this paper. Further, the BS-BHM model that it is improved the result of Gaussian integral, especially in completing square is named BS-BHM Updated model. Both BS-BHM model and BS-BHM Updated model have lognormal distribution. BS-BHM model and BS-BHM Updated model also have the same as two parameter that is the volatility parameter ν and the information flow rate parameter σ. Estimation of the two parameters result an equation of polynomial of four degree under BS-BHM model and rerult a quadratic equation under BS-BHM Updated model. Application for real data of Microsoft shares results four value of estimation of parameters, while under BS-BHM Updated model result one value of estimation of parameter. All about BS-BHM Updated model can be seen Mutijah,Guritno, S, and Gunardi [7, 8].

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